ON COHOMOLOGICAL OBSTRUCTIONS TO THE EXISTENCE OF B-LOG SYMPLECTIC STRUCTURES

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Abstract. We prove that a compact b-log symplectic manifold has a class in the second cohomology group whose powers, except maybe for the top, are nontrivial. This result gives cohomological obstructions for the existence of b-log symplectic structures similar to those in symplectic geometry.

A Poisson structure π on a smooth manifold M of dimension $2n$ is called a *b-log symplectic* structure if the map

$$\wedge^n\pi : M \longrightarrow \Lambda^{2n}TM, \quad x \mapsto \wedge^n\pi(x)$$

is transverse to the zero section. This class of Poisson structures was introduced on 2-dimensional surfaces in [3] (under the name of *topologically stable Poisson structures*) where a complete classification was obtained; in higher dimensions a systematic investigation of the geometric properties of such structures appeared in [2] (where these are called *b-Poisson* and *b-symplectic* structures); and their integrations by symplectic groupoids were studied in [1] (where they are called *log symplectic* structures).

Our interest in b-log symplectic structures comes from the fact that these can be used to construct regular corank-one Poisson structures. First, the singular locus of a b-log symplectic structure $Z := (\wedge^n\pi)^{-1}(0)$ (if nonempty) carries a regular corank-one Poisson structure with a very special property: it has a transverse Poisson vector field [2]. Secondly, a b-log structure can be used to construct a regular corank-one Poisson on $M \times S^1$, simply given by

$$\pi + X \wedge \frac{\partial}{\partial\theta},$$

where X is the modular vector field of (M,π). However, our result excludes the possibility of using this procedure to construct corank-one Poisson structures in some interesting examples, e.g. on $S^4 \times S^1$.

Our result is the following:

Theorem. *Let (M^{2n},π) be a compact b-log symplectic manifold. Then there exists a class $c \in H^2(M)$ such that $c^{n-1} \in H^{2n-2}(M)$ is nonzero.*

Proof. Denote by $Z := (\wedge^n\pi)^{-1}(0)$ the singular locus of π. If $Z = \emptyset$, we can apply the usual argument from symplectic geometry. Assume then $Z \neq \emptyset$.

We will first assume that M is orientable. Let μ be a volume form on M and denote by $t := \langle\pi^n,\mu\rangle$. The singular locus is $Z = \{t = 0\}$ and the b-log condition implies that t is a submersion along Z, so we can find a retraction $r : U \to Z$, where U is an open around Z, such that $(r,t) : U \xrightarrow{\sim} Z \times (-\delta,\delta)$ is a diffeomorphism. Since Z is a Poisson submanifold (it is fixed by all Poisson automorphisms, hence all Hamiltonians are tangent to Z), in this open we can write $\pi = t\partial/\partial t \wedge X_t + w_t$ for a vector field X_t and a bivector w_t on Z, both depending smoothly on t. Since $1/t\pi^n = n\partial/\partial t \wedge X_t \wedge w_t^{n-1}$ is nowhere vanishing, we have that the bivector $\partial/\partial t \wedge X_t + w_t$ is invertible. Denote its inverse by $\alpha_t \wedge dt + \beta_t$, with α_t and β_t forms on Z depending smoothly on $t \in (-\delta,\delta)$. Then $\omega := \pi|_{M\setminus Z}^{-1}$, the inverse of π, can be

written as

$$\omega|_{U\setminus Z} = \alpha_t \wedge dt/t + \beta_t.$$

Since ω is closed we get that α_0 and β_0 are closed, and since $dt\wedge\alpha_0+\beta_0$ is invertible, it follows that $\alpha_0 \wedge \beta_0^{n-1}$ is a volume form on Z. Since Z is compact, this implies that β_0^{n-1} cannot be exact. We will construct a closed 2-form ω' on M whose pullback to Z is β_0; hence $c := [\omega']$ will satisfy the conclusion of the theorem.

Let $\chi : (-\delta, \delta) \to \mathbb{R}$ be a bump function that takes the value 1 for $|t| \leq \delta/4$, and 0 for $|t| \geq \delta/2$. Consider the 2-form ω' on $M\setminus Z$ that coincides with ω outside of U and on $U\setminus Z$ it is given by

$$\omega'|_{U\setminus Z} = (\alpha_t - \chi(t)\alpha_0) \wedge dt/t + \beta_t.$$

ω' extends smoothly to Z, since for $|t| \leq \delta/4$ it can be written as $\omega' = \lambda_t \wedge dt + \beta_t$, where $\lambda_t = \int_0^1 \dot{\alpha}_{ts} ds$, or equivalently $\alpha_t = \alpha_0 + t\lambda_t$. So ω' is a closed 2-form on M whose pullback to Z is β_0; thus $[\omega']^{n-1} \neq 0$.

If M is not orientable, consider $p : \widetilde{M} \to M$ the orientable double cover, and let $\gamma : \widetilde{M} \xrightarrow{\simeq} \widetilde{M}$ be the corresponding deck transformation. We first construct a tubular neighborhood $(\widetilde{r}, t) : \widetilde{U} \xrightarrow{\simeq} \widetilde{Z} \times (-\delta, \delta)$ of the singular locus $\widetilde{Z} := p^{-1}(Z)$ of $\widetilde{\pi} := p^*(\pi)$, with $\widetilde{U} = p^{-1}(U)$, and such that the action of γ corresponds to $\gamma(z, t) = (\gamma(z), -t)$, for $(z, t) \in \widetilde{Z} \times (-\delta, \delta)$. The map $\widetilde{r} : \widetilde{U} \to \widetilde{Z}$ can be constructed by lifting a retraction $r : U \to Z$. Consider a volume form μ_0, and denote by f the smooth function satisfying $\gamma^*(\mu_0) = -e^f \mu_0$. Then the volume form $\mu := e^{f/2}\mu_0$ satisfies $\gamma^*(\mu) = -\mu$. Thus, by shrinking U, we can use $t := \langle \widetilde{\pi}^n, \mu \rangle$ to construct the desired tubular neighborhood. As before, on $\widetilde{Z} \times (-\delta, \delta)$ we can write $p^*(\omega|_{U\setminus Z}) = \alpha_t \wedge dt/t + \beta_t$. Invariance under γ implies that $(\gamma|_{\widetilde{Z}})^*(\alpha_t) = \alpha_{-t}$ and $(\gamma|_{\widetilde{Z}})^*(\beta_t) = \beta_{-t}$. In particular α_0 and β_0 are invariant. Thus, choosing the function $\chi(t)$ from the construction from the orientable case to satisfy $\chi(t) = \chi(-t)$, we obtain an invariant closed 2-form ω' on $\widetilde{M}$ that satisfies $[\omega']^{n-1} \neq 0$. Invariance implies that $\omega' = p^*(\omega'')$ for a closed 2-form ω'' on M; hence $c := [\omega'']$ satisfies the conclusion.

□

Remark. Observe that for $Z \neq \emptyset$ the proof of the theorem uses only the compactness of Z and not that of M.

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